Stochastic Approximation of Network Reliability

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NAS

COMPLEX NETWORKS

Network Reliability

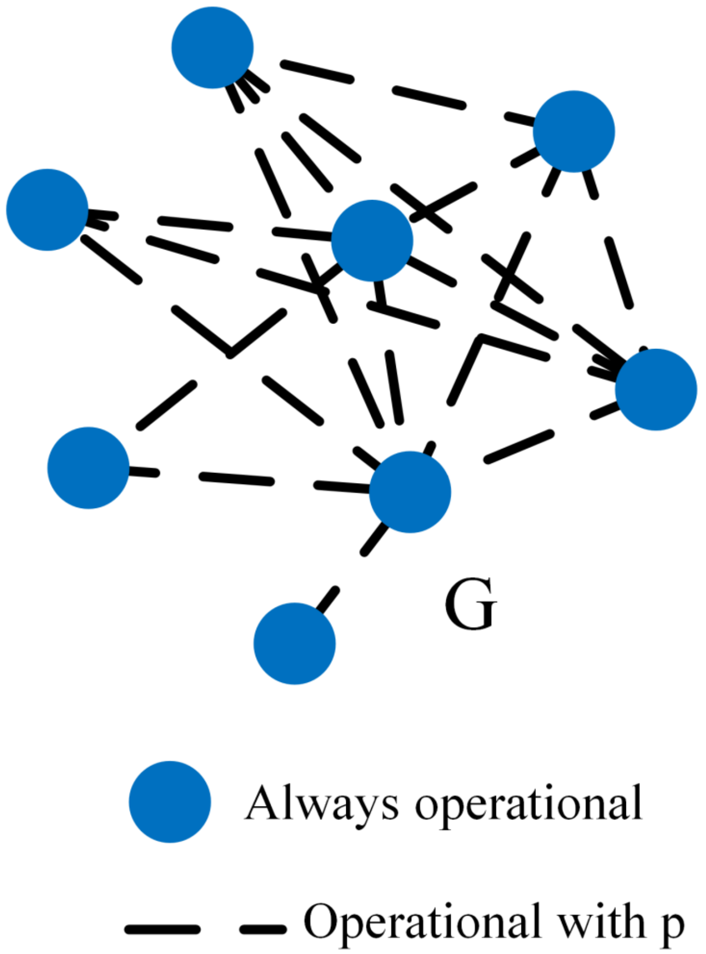
- Network Reliability

  Undirected graph $G(N, L)$

  Each link independently operational with probability p

  Nodes always operational

- Network Reliability = Pr[G is connected]

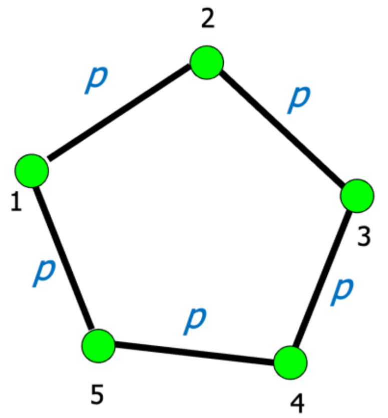


Network Reliability Polynomial $Rel_G(p)$

$$Rel_G(p) = \sum_{i=0}^{L-N+1} F_i (1-p)^i p^{L-i}$$

F_i: # of sets of i links, whose removal leave G connected

NP-hard

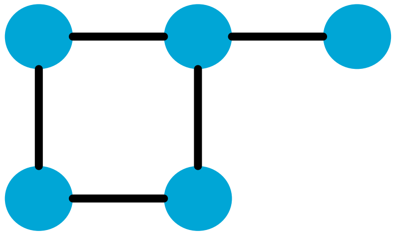


$F_0 = 1$
$F_1 = 5$

$$Rel_G(p) = p^5 + 5p^4(1-p)$$

Stochastic approximation of $Rel_G(p)$

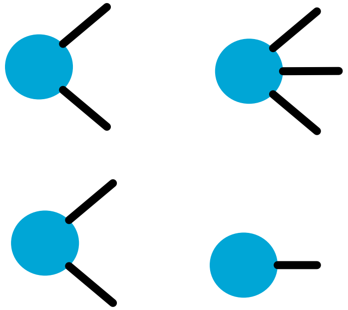
$\{G_{p_l}(N)$ is connected $\} \Longrightarrow \{D_{min} \geq 1\}$: always true



Connected Graph

$\longrightarrow$ $D_{min} > 0$

Main assumption: $\{D_{min} \geq 1\} \Longrightarrow \{G_{p_l}(N)$ is connected $\}$ for large N and p_l



$D_{min} > 0$

$\overset{?}{\longrightarrow}$ Connected Graph

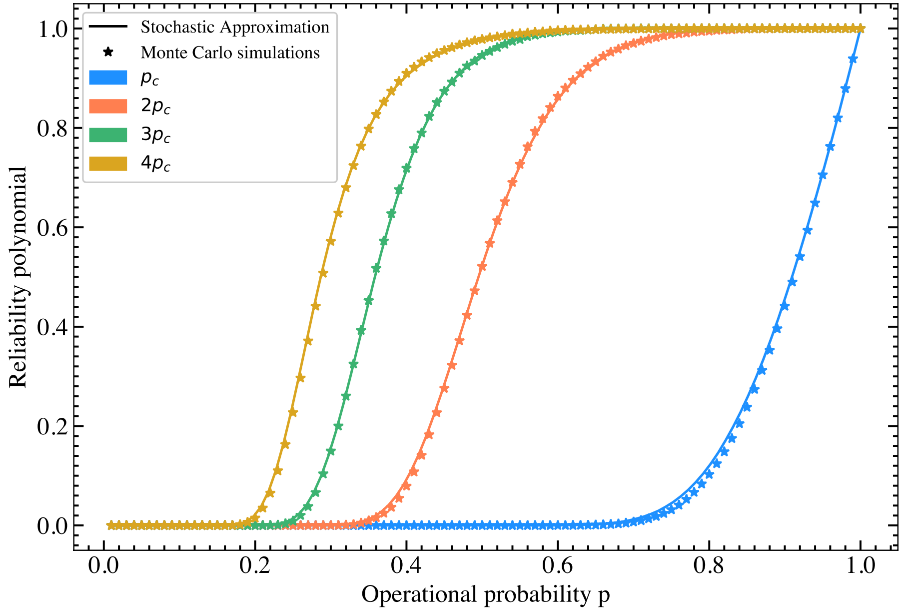
Stochastic approximation of $Rel_G(p)$

$\Pr[G \text{ is connected}] = \Pr[D_{min} \geq 1] + o(1)$

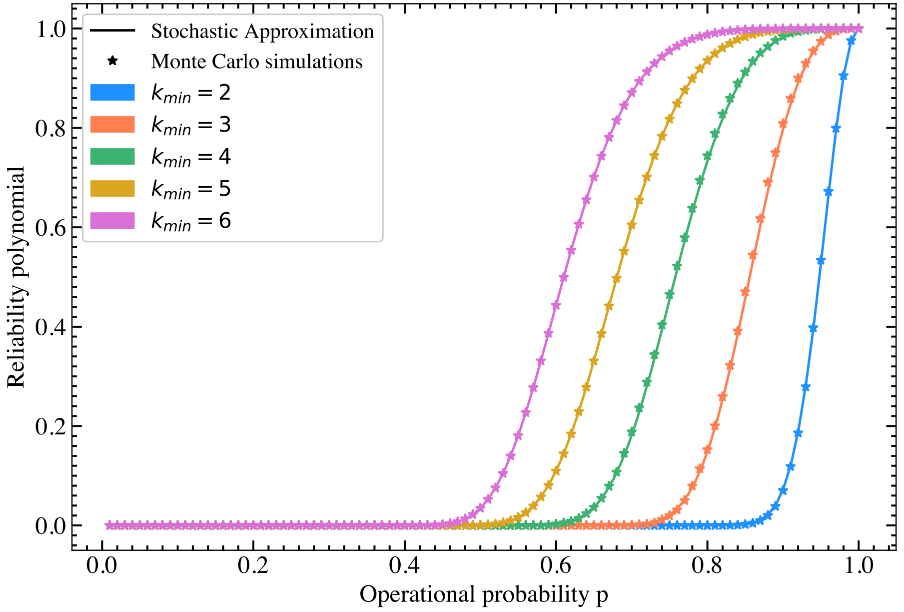
Stochastic approximation: $Rel_G(p) \approx \overline{Rel}_G(p) = (1 - \varphi_D(1-p))^N$

Where $\varphi_D(z) = E[z^D] = \sum_{j=0}^{N-1} \Pr[D = j] z^j$

Stochastic approximation of $Rel_G(p)$



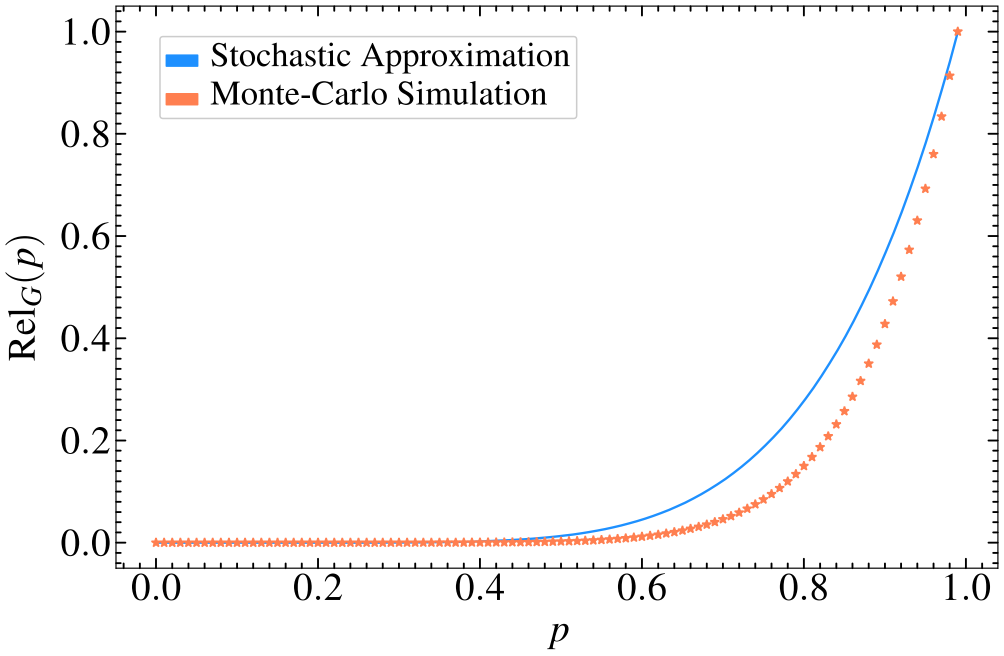
ER graphs with N=200, different link density

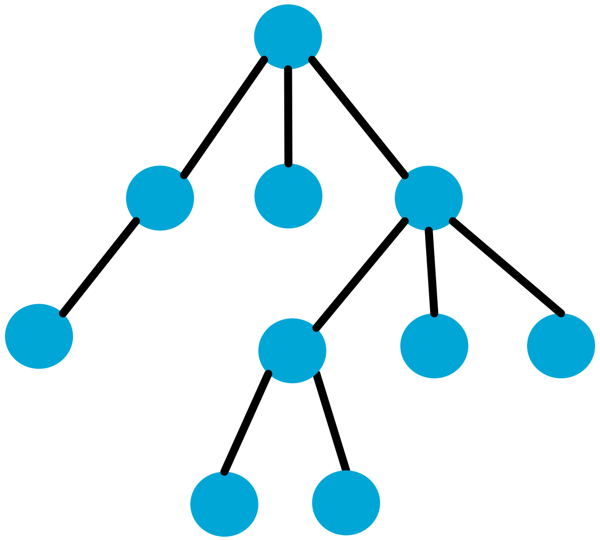


BA model with N=500, different numbers of edges per new node k_{min}

Accurate and based solely on degree distribution

Stochastic approximation of $Rel_G(p)$





Tree graph with $N = 10$ nodes and L=9 links

Work bad for small N and p_l

Node reliability polynomial $nRel_G(p)$

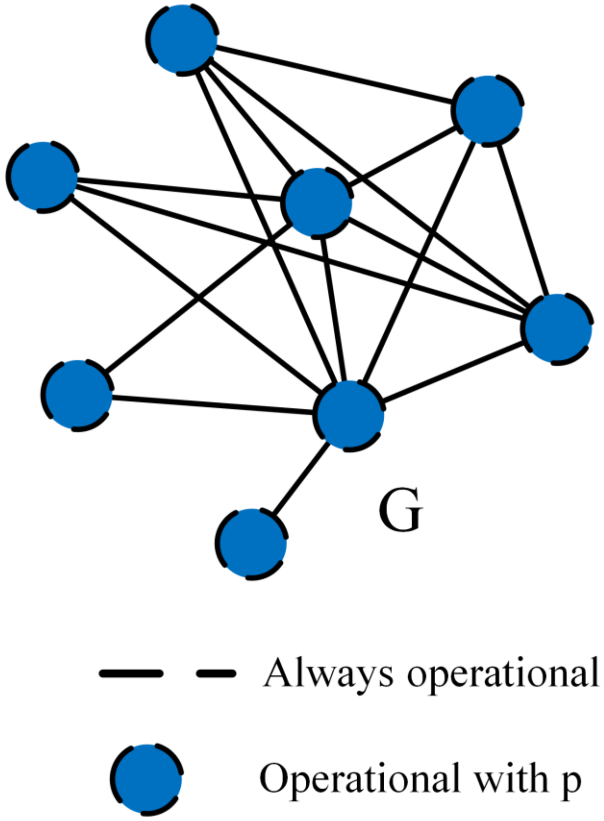
- Network Reliability

Undirected graph $G(N, L)$

Each node independently operational with probability p

Links always operational

- Network Reliability = Pr[G is connected]



Node reliability polynomial $nRel_G(p)$

$$nRel_G(p) = \sum_{k=0}^{N} S_k (1-p)^{N-k} p^N$$

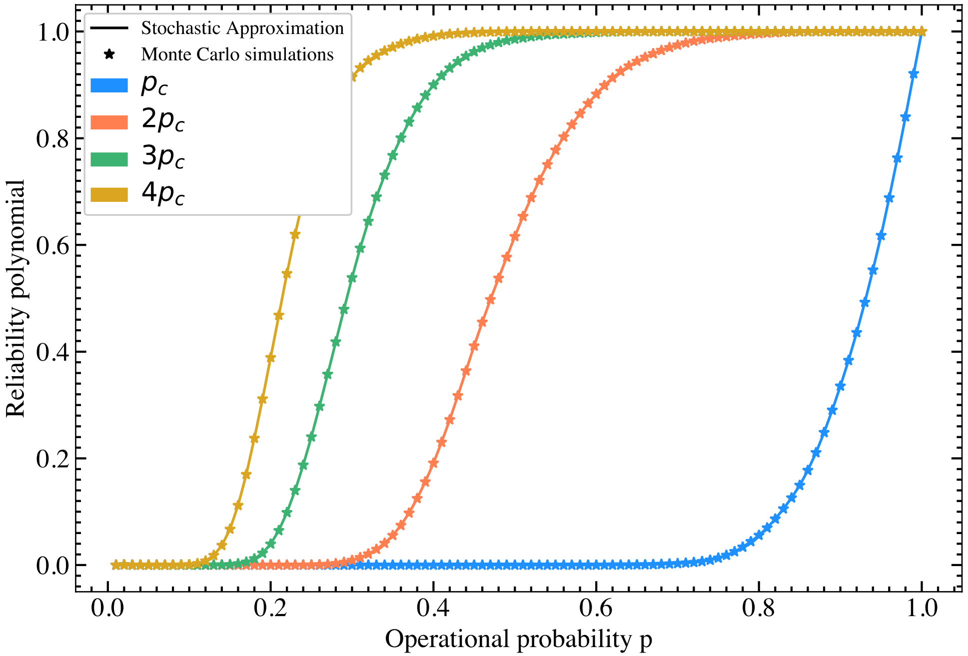
S_k : # of sets of connected subgraph of G with k nodes

NP-hard

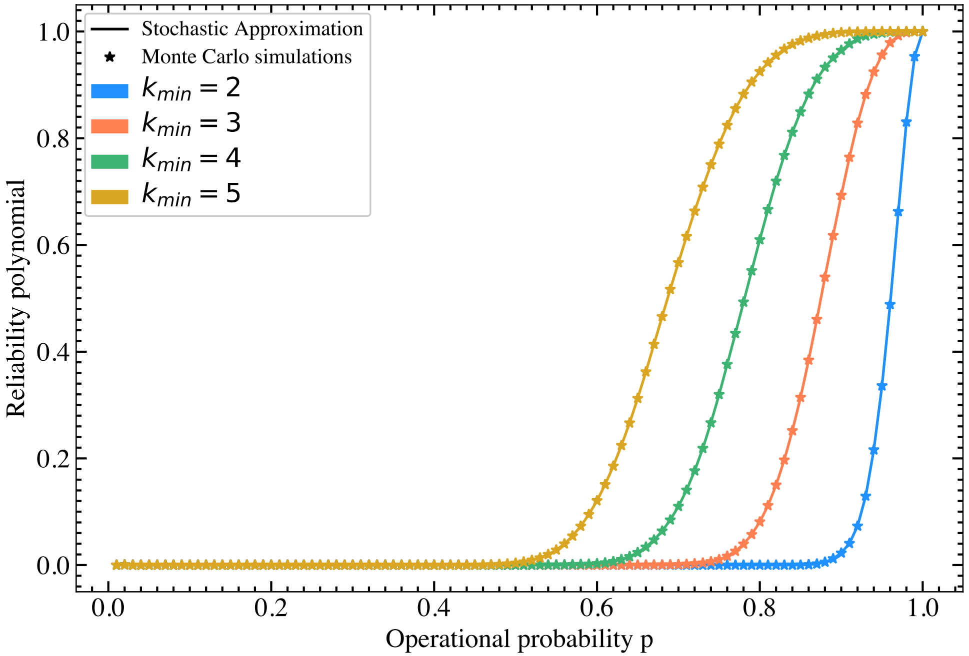
Stochastic approximation: $\mathrm{n}Rel_G(p) \approx \overline{nRel}_G(p) = (1 - \varphi_D(1-p))^{Np}$

Where $\varphi_D(z) = E[z^D] = \sum_{j=0}^{N-1} \Pr[D=j] z^j$

Stochastic approximation of $nRel_G(p)$

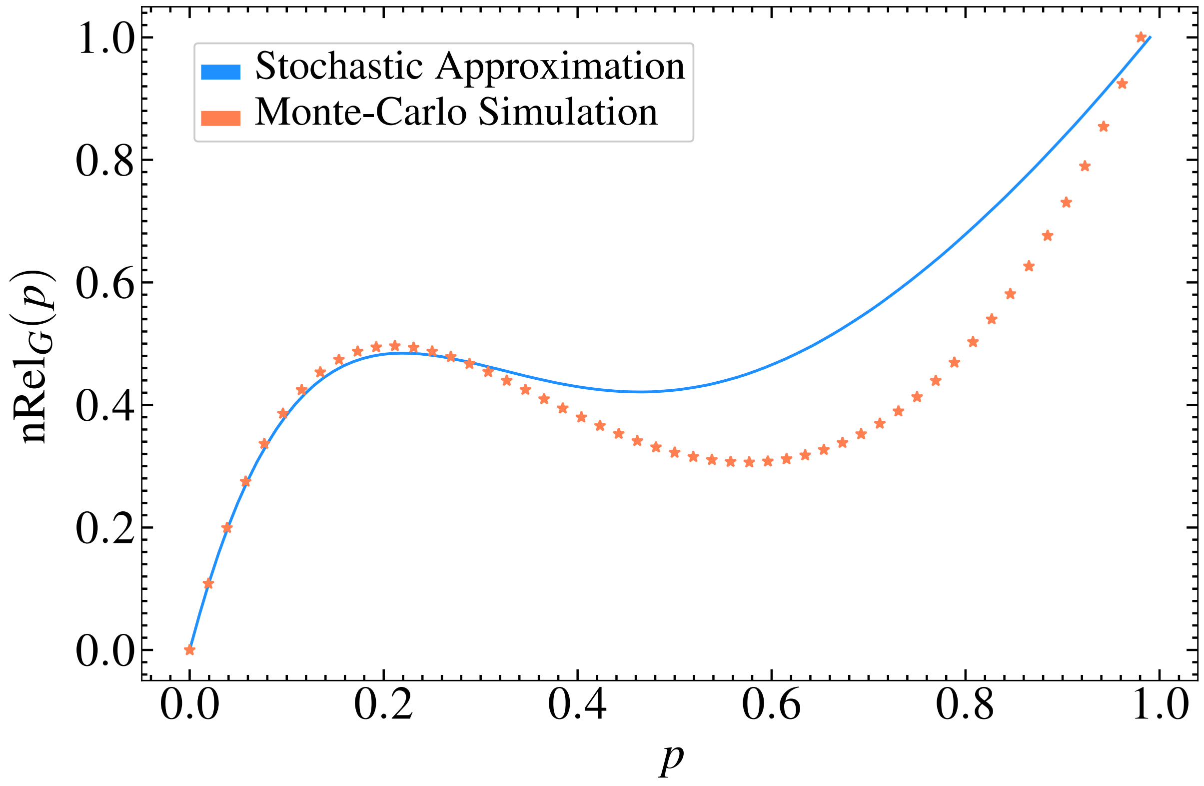


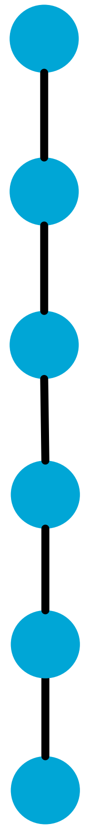
ER graphs with N=200, different link density



BA model with N=500, different numbers of edges per new node k_{min}

Stochastic approximation of $nRel_G(p)$

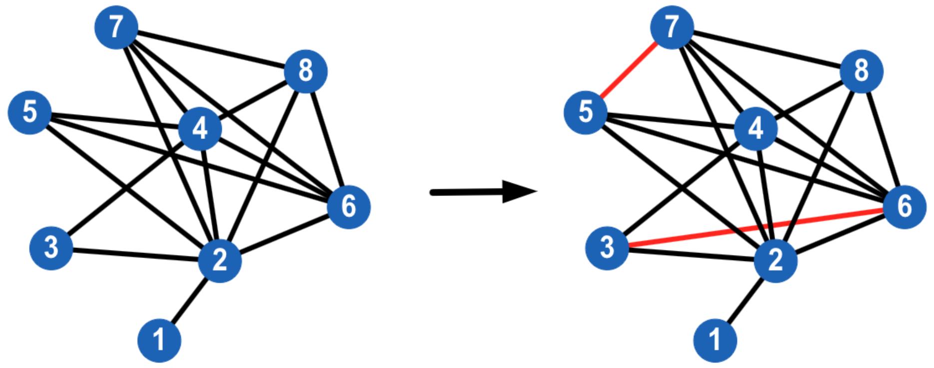




Work bad for small N and p_l

Enhancing Network Reliability by Adding l Edges

Adding l links to maximize the network reliability $Rel_G(p)$ or node reliability $\mathrm{n}Rel_G(p)$



NP-hard

Reliability based k-GRIP problem

$$Rel_G(p) \approx (1 - \varphi(1-p))^N$$

$$nRel_G(p) \approx (1 - \varphi(1-p))^{Np}$$

Depend on $1 - \varphi(1-p)$

$$1 - \varphi(1-p) = \frac{1}{N}\sum_{i=1}^{N}(1-(1-p)^{d_i})$$

Objective:

$$\max_A 1 - \varphi_{D+A}(1-p)$$

$$= \max_{A=[a_1,a_2,\ldots,a_N]} \sum_{i=1}^{N}\left(1-(1-p)^{d_i+a_i}\right)$$

Subject to:

$$s.t. \sum_{i=1}^{N} a_i = 2k, a_i \geq 0, a_i \in \mathbb{Z}$$

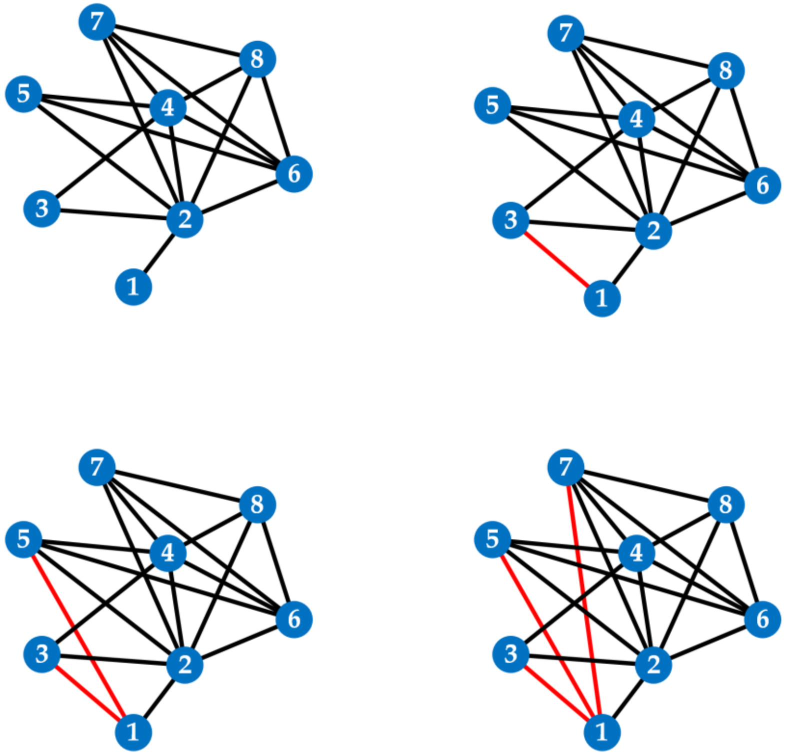
Greedily add links between nodes with the lowest degrees

Greedy Algorithm

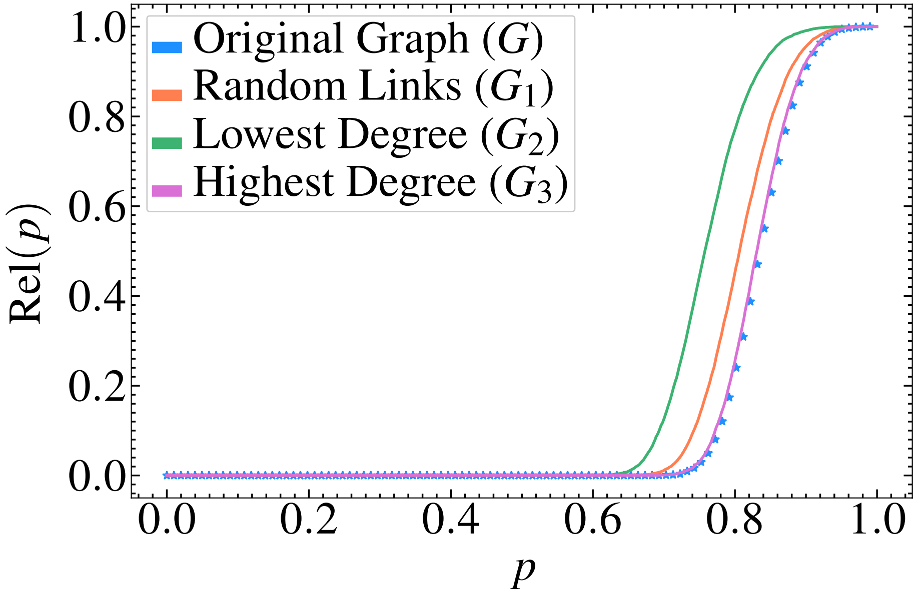
Algorithm 1 Greedy Lowest-Degree Pairing Edge Addition Algorithm

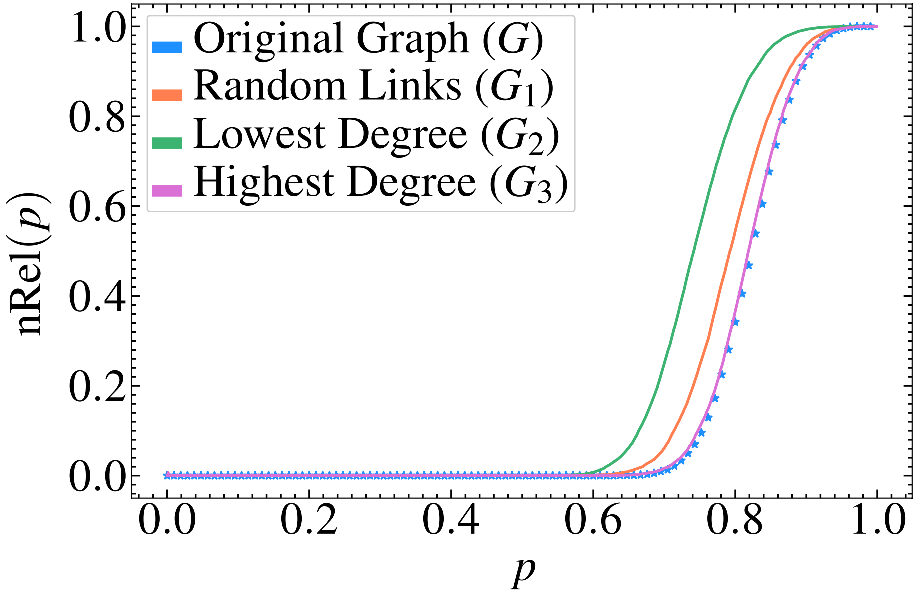
Input: a graph G, number of links to add k
Output: a new graph G^*

Generate the degree vector $\mathbf{d}$ for graph G
for $t = 1$ to k **do**
  Sort nodes by their degree in ascending order
  Find node i with the smallest degree
  Find node j with the smallest degree that is not connected to i
  Add link between nodes i and j in the graph
  Update the graph G and the degree vector $\mathbf{d}$ after adding the new link
end for
Return the new graph G^*



Greedy Algorithm





N = 1365 nodes, L = 5263 links, 500 links are added

Thank You

Paper: Node Reliability: Approximation, Upper Bounds, and Applications to Network Robustness

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