Controllability of Networks Under Sparsity Constraints

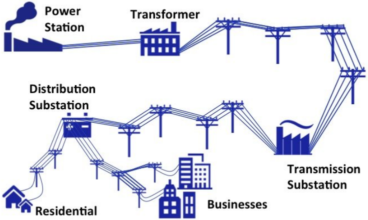
Geethu Joseph

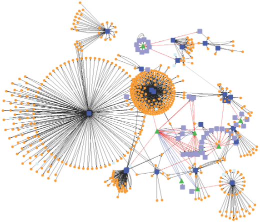
Signal Processing Systems Group

Seminar Graphs&Data@TU Delft

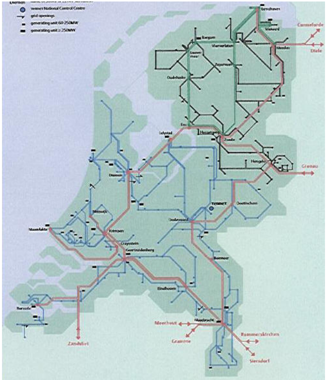
The Invisible Webs Around Us

- Several "controllable" network systems keep our society functioning
  e.g., social, biological, chemical, energy, financial, defense, healthcare, etc.

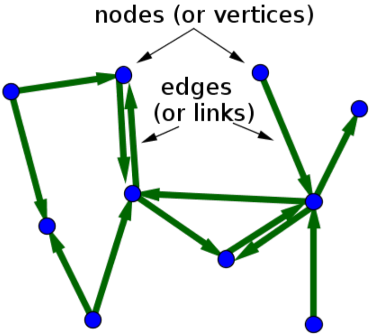




Network systems are realized through interconnected subsystems



Map of Dutch Electricity Grid



Control theory: analyzing and influencing the network nodes to change the system behavior

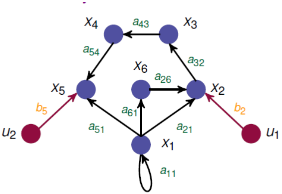
State Space Model

State Evolution: $\boldsymbol{x}_k = \boldsymbol{A}\boldsymbol{x}_{k-1} + \boldsymbol{B}\boldsymbol{u}_k$

Observations: $\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{w}_k$

State vector	$\boldsymbol{x}_k \in \mathbb{R}^{N\times 1}$	Transition matrix	$\boldsymbol{A} \in \mathbb{R}^{N\times N}$
Measurement vector	$\boldsymbol{y}_k \in \mathbb{R}^{m\times 1}$	Input matrix	$\boldsymbol{B} \in \mathbb{R}^{N\times L}$
Measurement noise	$\boldsymbol{w}_k \in \mathbb{R}^{m\times 1}$	Output matrix	$\boldsymbol{C} \in \mathbb{R}^{m\times N}$

$$\boldsymbol{x}_k = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 & 0 & a_{26} \\ 0 & a_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{43} & 0 & 0 & 0 \\ a_{51} & 0 & 0 & a_{54} & 0 & 0 \\ a_{61} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}_{k-1} + \begin{bmatrix} 0 & 0 \\ b_2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & b_5 \\ 0 & 0 \end{bmatrix} \boldsymbol{u}_k$$



$$\text{State Evolution: } \boldsymbol{x}_k = \boldsymbol{A}\boldsymbol{x}_{k-1} + \boldsymbol{B}\boldsymbol{u}_k$$

$$\text{Observations: } \boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{w}_k$$

State	Social opinion (e.g., movie rating, political inclination)
Transition matrix	Social connections (e.g.: Instagram/Twitter followers, neighbors)
Input	Social influencers (e.g.: marketing agents, political leaders)
Observations	Rating systems (e.g.: IMDb ratings, gross sales, surveys)

What is Sparse Control?

Sparse Control is minimal intervention!

Why sparse control?

- Biological systems: Minimal drug control
- Resource-constrained system: Low communication and computational burden
- Social network: Budget-constrained advertising
- Cyber-physical attacks: Limited access to the system

What is Sparse Control?

Sparse Control is minimal intervention!

Why sparse control?

- Biological systems: Minimal drug control
- Resource-constrained system: Low communication and computational burden
- Social network: Budget-constrained advertising
- Cyber-physical attacks: Limited access to the system

- Control vector with a lot of zeros
- Uses only a few actuators among the available ones
- Admits compact representations (thanks to compressed sensing)

This talk: Sparse control in discrete-time linear dynamical systems

Talk Outline

- **Feasibility of sparse control:** When does sparse control work?
- **Optimal control:** How do we design/detect sparse control?
- **Case study:** Social network manipulation

- Control vector with a lot of zeros
- Uses only a few actuators among the available ones
- Admits compact representations (thanks to compressed sensing)

This talk: Sparse control in discrete-time linear dynamical systems

Talk Outline

- Feasibility of sparse control: When does sparse control work?
- Optimal control: How do we design/detect sparse control?
- Case study: Social network manipulation

- Control vector with a lot of zeros
- Uses only a few actuators among the available ones
- Admits compact representations (thanks to compressed sensing)

This talk: Sparse control in discrete-time linear dynamical systems

Talk Outline

- **Feasibility of sparse control:** When does sparse control work?
- **Optimal control:** How do we design/detect sparse control?
- **Case study:** Social network manipulation

What is Sparsity?

- Control vector with a lot of zeros
- Uses only a few actuators among the available ones
- Admits compact representations (thanks to compressed sensing)

This talk: Sparse control in discrete-time linear dynamical systems

Talk Outline

- **Feasibility of sparse control:** When does sparse control work?
- **Optimal control:** How do we design/detect sparse control?
- **Case study:** Social network manipulation

Sparsity-constrained Linear Dynamical System

State Space Model

State Evolution: $\boldsymbol{x}_k = \boldsymbol{A}\boldsymbol{x}_{k-1} + \boldsymbol{B}\boldsymbol{u}_k$

Observations: $\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{w}_k$

State vector	$\boldsymbol{x}_k \in \mathbb{R}^{N\times 1}$	Transition matrix	$\boldsymbol{A} \in \mathbb{R}^{N\times N}$
Measurement vector	$\boldsymbol{y}_k \in \mathbb{R}^{m\times 1}$	Input matrix	$\boldsymbol{B} \in \mathbb{R}^{N\times L}$
Measurement noise	$\boldsymbol{w}_k \in \mathbb{R}^{m\times 1}$	Output matrix	$\boldsymbol{C} \in \mathbb{R}^{m\times N}$

Constraint

Sparsity: $\|\boldsymbol{u}_k\|_0 \ll L, \quad k = 1, 2, \ldots$

Time invariant or time varying support

Controllability Using Sparse Inputs

Ability of a control system to reach any given state from any initial state in a finite time

$$\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{B} \boldsymbol{x}_{\text{init}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_K \end{bmatrix}$$

Two classic tests for controllability without any constraints:

1. **Kalman test:**

$$\text{Rank}\left\{\begin{bmatrix} \boldsymbol{A}^{N-1}\boldsymbol{B} & \boldsymbol{A}^{N-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}\right\} = N$$

2. **Popov–Belevitch–Hautus test (PBH test):**

$$\text{Rank}\left\{\begin{bmatrix} \boldsymbol{A} - \lambda \boldsymbol{I} & \boldsymbol{B} \end{bmatrix}\right\} = N \quad \forall \lambda \in \mathbb{C}$$

Sparse Controllability: Time Invariant Support

$$\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{B}\boldsymbol{x}_{\text{init}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_K \end{bmatrix}$$

- Assume that the support of $\boldsymbol{u}_k$ is $\mathcal{S} \subseteq \{1, 2, \ldots, N\}$, for all values of k

$$\exists K < \infty \text{ such that } \boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{B}\boldsymbol{x}_{\text{init}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}} & \ldots \boldsymbol{B}_{\mathcal{S}} \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} \boldsymbol{u}_{1,\mathcal{S}} \\ \boldsymbol{u}_{2,\mathcal{S}} \\ \vdots \\ \boldsymbol{u}_{K,\mathcal{S}} \end{bmatrix}$$

Equivalent Condition: PBH Test for $(\boldsymbol{A}, \boldsymbol{B}_{\mathcal{S}})$

There exists $K < \infty$ and $\mathcal{S}$ with $|\mathcal{S}| = s$

$$\text{Rank}\left\{\begin{bmatrix} \boldsymbol{A} - \lambda \boldsymbol{I} & \boldsymbol{B}_{\mathcal{S}} \end{bmatrix}\right\} = N \quad \forall \lambda \in \mathbb{C}$$

NP hard complexity!

Sparse Controllability: Time Invariant Support

$$\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{B} \boldsymbol{x}_{\text{init}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_K \end{bmatrix}$$

- Assume that the support of $\boldsymbol{u}_k$ is $\mathcal{S} \subseteq \{1, 2, \ldots, N\}$, for all values of k

$$\exists K < \infty \text{ such that } \boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{B} \boldsymbol{x}_{\text{init}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}} & \ldots \boldsymbol{B}_{\mathcal{S}} \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} \boldsymbol{u}_{1,\mathcal{S}} \\ \boldsymbol{u}_{2,\mathcal{S}} \\ \vdots \\ \boldsymbol{u}_{K,\mathcal{S}} \end{bmatrix}$$

Equivalent Condition: PBH Test for $(\boldsymbol{A}, \boldsymbol{B}_{\mathcal{S}})$

There exists $K < \infty$ and $\mathcal{S}$ with $|\mathcal{S}| = s$

$$\text{Rank}\left\{\begin{bmatrix} \boldsymbol{A} - \lambda \boldsymbol{I} & \boldsymbol{B}_{\mathcal{S}} \end{bmatrix}\right\} = N \quad \forall \lambda \in \mathbb{C}$$

NP hard complexity!

Sparse Controllability: Time Invariant Support

$$\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{B} \boldsymbol{x}_{\text{init}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_K \end{bmatrix}$$

- Assume that the support of $\boldsymbol{u}_k$ is $\mathcal{S} \subseteq \{1, 2, \ldots, N\}$, for all values of k

$$\exists K < \infty \text{ such that } \boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{B} \boldsymbol{x}_{\text{init}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}} & \ldots \boldsymbol{B}_{\mathcal{S}} \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} \boldsymbol{u}_{1,\mathcal{S}} \\ \boldsymbol{u}_{2,\mathcal{S}} \\ \vdots \\ \boldsymbol{u}_{K,\mathcal{S}} \end{bmatrix}$$

Equivalent Condition: PBH Test for $(\boldsymbol{A}, \boldsymbol{B}_{\mathcal{S}})$

There exists $K < \infty$ and $\mathcal{S}$ with $|\mathcal{S}| = s$

$$\text{Rank}\left\{\begin{bmatrix} \boldsymbol{A} - \lambda \boldsymbol{I} & \boldsymbol{B}_{\mathcal{S}} \end{bmatrix}\right\} = N \quad \forall \lambda \in \mathbb{C}$$

NP hard complexity!

Sparse Controllability: Time Varying Support

$$\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{B} \boldsymbol{x}_{\text{init}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_K \end{bmatrix}$$

- Assume that the support of $\boldsymbol{u}_k$ is $\mathcal{S}_k \subseteq \{1, 2, \ldots, N\}$

$$\exists K < \infty \text{ such that } \mathbb{R}^N = \underbrace{\bigcup_{|\mathcal{S}_k| \leq s} \text{Span}\left\{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2} & \ldots \boldsymbol{B}_{\mathcal{S}_{K-1}} \end{bmatrix}\right\}}_{\text{finite union of subspaces}}$$

Equivalent Condition: Kalman rank-type Test

There exists $K < \infty$ and $\{\mathcal{S}_k\}_{k=1}^K$ with $|\mathcal{S}_k| = s$

$$\text{Rank}\left\{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2} & \ldots \boldsymbol{B}_{\mathcal{S}_K} \end{bmatrix}\right\} = N$$

Question: can we find a simpler sparse controllability condition?

Sparse Controllability: Time Varying Support

$$\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{B} \boldsymbol{x}_{\text{init}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_K \end{bmatrix}$$

- Assume that the support of $\boldsymbol{u}_k$ is $\mathcal{S}_k \subseteq \{1, 2, \ldots, N\}$

$$\exists K < \infty \text{ such that } \mathbb{R}^N = \underbrace{\bigcup_{|\mathcal{S}_k| \leq s} \operatorname{Span}\left\{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2} & \ldots \boldsymbol{B}_{\mathcal{S}_{K-1}} \end{bmatrix}\right\}}_{\text{finite union of subspaces}}$$

Equivalent Condition: Kalman rank-type Test

There exists $K < \infty$ and $\{\mathcal{S}_k\}_{k=1}^K$ with $|\mathcal{S}_k| = s$

$$\operatorname{Rank}\left\{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2} & \ldots \boldsymbol{B}_{\mathcal{S}_K} \end{bmatrix}\right\} = N$$

Question: can we find a simpler sparse controllability condition?

Sparse Controllability: Time Varying Support

$$\boldsymbol{x}_{\mathrm{final}} - \boldsymbol{A}^K \boldsymbol{B} \boldsymbol{x}_{\mathrm{init}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\text{controllability matrix}} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_K \end{bmatrix}$$

- Assume that the support of $\boldsymbol{u}_k$ is $\mathcal{S}_k \subseteq \{1, 2, \ldots, N\}$

$$\exists K < \infty \text{ such that } \mathbb{R}^N = \underbrace{\bigcup_{|\mathcal{S}_k| \leq s} \mathrm{Span}\left\{ \begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2} & \ldots \boldsymbol{B}_{\mathcal{S}_{K-1}} \end{bmatrix} \right\}}_{\text{finite union of subspaces}}$$

Equivalent Condition: Kalman rank-type Test

There exists $K < \infty$ and $\{\mathcal{S}_k\}_{k=1}^K$ with $|\mathcal{S}_k| = s$

$$\mathrm{Rank}\left\{ \begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2} & \ldots \boldsymbol{B}_{\mathcal{S}_K} \end{bmatrix} \right\} = N$$

Question: can we find a simpler sparse controllability condition?

Necessary Conditions

1. Sparse controllability $\implies$ controllability $\implies$ PBH test succeeds

2. Kalman rank-type test succeeds $\implies \exists K$ and $\{\mathcal{S}_k\}_{k=1}^{K}$ such that

$$\text{Rank}\left\{\begin{bmatrix}\boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2} & \dots & \boldsymbol{A}\boldsymbol{B}_{\mathcal{S}_{K-1}} & \boldsymbol{B}_{\mathcal{S}_K}\end{bmatrix}\right\} = N$$

$$\text{Rank}\left\{\begin{bmatrix}\underbrace{\boldsymbol{A}\begin{bmatrix}\boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-3}\boldsymbol{B}_{\mathcal{S}_2} & \dots & \boldsymbol{B}_{\mathcal{S}_{K-1}}\end{bmatrix}}_{\text{subset of span of } \boldsymbol{A}} & \boldsymbol{B}_{\mathcal{S}_K}\end{bmatrix}\right\} = N$$

$$\implies \text{Rank}\left\{\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}_{\mathcal{S}_K}\end{bmatrix}\right\} = N$$

$$\implies \text{Rank}\left\{\boldsymbol{A}\right\} + s \geq N$$

Necessary conditions

1. For all $\lambda \in \mathbb{C}$, $\text{Rank}\left\{\begin{bmatrix}\boldsymbol{A} - \lambda\boldsymbol{I} & \boldsymbol{B}\end{bmatrix}\right\} = N$
2. $s \geq N - \text{Rank}\left\{\boldsymbol{A}\right\}$

These are also sufficient for $s-$sparse controllability!

Necessary Conditions

1. Sparse controllability $\implies$ controllability $\implies$ PBH test succeeds

2. Kalman rank-type test succeeds $\implies$ $\exists K$ and $\{\mathcal{S}_k\}_{k=1}^{K}$ such that

$$\text{Rank}\left\{\begin{bmatrix}\boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2} & \ldots & \boldsymbol{A}\boldsymbol{B}_{\mathcal{S}_{K-1}} & \boldsymbol{B}_{\mathcal{S}_K}\end{bmatrix}\right\} = N$$

$$\text{Rank}\left\{\begin{bmatrix}\underbrace{\boldsymbol{A}\begin{bmatrix}\boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-3}\boldsymbol{B}_{\mathcal{S}_2} & \ldots & \boldsymbol{B}_{\mathcal{S}_{K-1}}\end{bmatrix}}_{\text{subset of span of } \boldsymbol{A}} & \boldsymbol{B}_{\mathcal{S}_K}\end{bmatrix}\right\} = N$$

$$\implies \text{Rank}\left\{\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}_{\mathcal{S}_K}\end{bmatrix}\right\} = N$$

$$\implies \text{Rank}\left\{\boldsymbol{A}\right\} + s \geq N$$

Necessary conditions

1. For all $\lambda \in \mathbb{C}$, $\text{Rank}\left\{\begin{bmatrix}\boldsymbol{A} - \lambda\boldsymbol{I} & \boldsymbol{B}\end{bmatrix}\right\} = N$
2. $s \geq N - \text{Rank}\left\{\boldsymbol{A}\right\}$

These are also sufficient for $s-$sparse controllability!

1. Sparse controllability $\implies$ controllability $\implies$ PBH test succeeds

2. Kalman rank-type test succeeds $\implies$ $\exists K$ and $\{\mathcal{S}_k\}_{k=1}^{K}$ such that

$$\text{Rank}\left\{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_2} & \ldots & \boldsymbol{A}\boldsymbol{B}_{\mathcal{S}_{K-1}} & \boldsymbol{B}_{\mathcal{S}_K}\end{bmatrix}\right\} = N$$

$$\text{Rank}\left\{\begin{bmatrix} \underbrace{\boldsymbol{A}\begin{bmatrix}\boldsymbol{A}^{K-2}\boldsymbol{B}_{\mathcal{S}_1} & \boldsymbol{A}^{K-3}\boldsymbol{B}_{\mathcal{S}_2} & \ldots & \boldsymbol{B}_{\mathcal{S}_{K-1}}\end{bmatrix}}_{\text{subset of span of } \boldsymbol{A}} & \boldsymbol{B}_{\mathcal{S}_K}\end{bmatrix}\right\} = N$$

$$\implies \text{Rank}\left\{\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}_{\mathcal{S}_K}\end{bmatrix}\right\} = N$$

$$\implies \text{Rank}\{\boldsymbol{A}\} + s \geq N$$

Necessary conditions

1. For all $\lambda \in \mathbb{C}$, $\text{Rank}\left\{\begin{bmatrix}\boldsymbol{A} - \lambda\boldsymbol{I} & \boldsymbol{B}\end{bmatrix}\right\} = N$
2. $s \geq N - \text{Rank}\{\boldsymbol{A}\}$

These are also sufficient for $s-$sparse controllability!

Sparse Controllability: Necessary and Sufficient Conditions

$$\boldsymbol{x}_k = \boldsymbol{A}\boldsymbol{x}_{k-1} + \boldsymbol{B}\boldsymbol{u}_k \text{ and } \|\boldsymbol{u}_k\|_0 \leq s,$$

s-sparse controllability $\iff$

1. For all $\lambda \in \mathbb{C}$, Rank $\left\{ \begin{bmatrix} \boldsymbol{A} - \lambda \boldsymbol{I} & \boldsymbol{B} \end{bmatrix} \right\} = N$
2. $s \geq N - \text{Rank}\{\boldsymbol{A}\}$

- Classical PBH test is a special case
- **Kalman decomposition-type procedure** separating state space:
  - Sparse-controllable + Sparse-uncontrollable + Uncontrollable states

Number of Sparse Control Inputs

Bounds on the **number of input vectors K^*** to guarantee s-sparse controllability

$$\frac{N}{R^*_{\boldsymbol{B},s}} \leq K^* \leq \min\left\{q\left\lceil\frac{S^*}{s}\right\rceil, N - R^*_{\boldsymbol{B},s} + 1\right\} \leq N,$$

$$R^*_{\boldsymbol{B},s} \triangleq \min\left\{\text{Rank}\left\{\boldsymbol{B}\right\}, s\right\}$$

$$q \triangleq \text{degree of the minimal polynomial of } \boldsymbol{A}$$

$$S^* \triangleq N - \text{Rank}\left\{\boldsymbol{A}\right\}$$

Other Extensions: Sparse Control Inputs

- **Sparse controllability - fixed support**
  $\Leftrightarrow$ **Rank** $\left\{\begin{bmatrix} \boldsymbol{A} - \lambda \boldsymbol{I} & \boldsymbol{B}_{\mathcal{S}} \end{bmatrix}\right\} = N \quad \forall \lambda \in \mathbb{C}$ **and some** $\mathcal{S}$ **with** $|\mathcal{S}| = s$
- **Sparse controllability**
  $\Leftrightarrow$ **Controllability and** $s \geq N -$ **Rank** $\{\boldsymbol{A}\}$
- Non-negative sparse controllability
  $\Leftrightarrow$ Non-negative controllability and $s \geq N -$ Rank $\{\boldsymbol{A}\}$
- Output sparse controllability
  $\Leftrightarrow$ Output controllability and bounds on sparsity
- Sparse stabilizability
  $\Leftrightarrow$ Stabilizability (no constraints on sparsity s)

- **Sparse controllability - fixed support**
  $\Leftrightarrow$ **Rank** $\left\{\begin{bmatrix} \boldsymbol{A}-\lambda\boldsymbol{I} & \boldsymbol{B}_{\mathcal{S}} \end{bmatrix}\right\} = N \quad \forall\lambda \in \mathbb{C}$ **and some** $\mathcal{S}$ **with** $|\mathcal{S}| = s$

- **Sparse controllability**
  $\Leftrightarrow$ **Controllability and** $s \geq N -$ **Rank** $\{\boldsymbol{A}\}$

- Non-negative sparse controllability
  $\Leftrightarrow$ Non-negative controllability and $s \geq N - \text{Rank}\{\boldsymbol{A}\}$

- Output sparse controllability
  $\Leftrightarrow$ Output controllability and bounds on sparsity

- Sparse stabilizability
  $\Leftrightarrow$ Stabilizability (no constraints on sparsity s)

Other Extensions: Sparse Control Inputs

- **Sparse controllability - fixed support**
  $\Leftrightarrow$ **Rank** $\left\{\begin{bmatrix} \boldsymbol{A} - \lambda \boldsymbol{I} & \boldsymbol{B}_{\mathcal{S}} \end{bmatrix}\right\} = N \quad \forall \lambda \in \mathbb{C}$ **and some** $\mathcal{S}$ **with** $|\mathcal{S}| = s$

- **Sparse controllability**
  $\Leftrightarrow$ **Controllability and** $s \geq N -$ **Rank** $\{\boldsymbol{A}\}$

- Non-negative sparse controllability
  $\Leftrightarrow$ Non-negative controllability and $s \geq N -$ Rank $\{\boldsymbol{A}\}$

- Output sparse controllability
  $\Leftrightarrow$ Output controllability and bounds on sparsity

- Sparse stabilizability
  $\Leftrightarrow$ Stabilizability (no constraints on sparsity s)

Other Extensions: Sparse Control Inputs

- **Sparse controllability - fixed support**
  $\Leftrightarrow$ **Rank** $\left\{\begin{bmatrix} \boldsymbol{A}-\lambda \boldsymbol{I} & \boldsymbol{B}_{\mathcal{S}} \end{bmatrix}\right\} = N \quad \forall \lambda \in \mathbb{C}$ **and some** $\mathcal{S}$ **with** $|\mathcal{S}| = s$

- **Sparse controllability**
  $\Leftrightarrow$ **Controllability and** $s \geq N -$ **Rank** $\{\boldsymbol{A}\}$

- Non-negative sparse controllability
  $\Leftrightarrow$ Non-negative controllability and $s \geq N - \text{Rank}\{\boldsymbol{A}\}$

- Output sparse controllability
  $\Leftrightarrow$ Output controllability and bounds on sparsity

- Sparse stabilizability
  $\Leftrightarrow$ Stabilizability (no constraints on sparsity s)

Case Study: Manipulation of Social Network

Who?
Marketing
Targeted fake-news campaigns
Political advertising

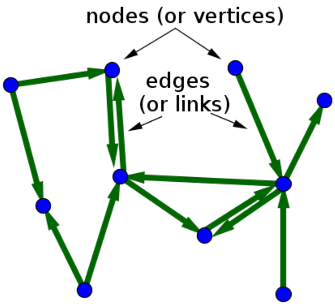
Why?
To control and change the public opinion in their favor

How?
By influencing a small number of people
due to budget/physical constraints

- Company offering free samples to a few individuals
- Election candidate visiting voters

Question: Is it possible to manipulate network opinion under budget constraints?

Image courtesy: ToughNickel, Romania Insider



Social network with N people

Weighted adjacency matrix:

$$\bar{\boldsymbol{A}} = \underbrace{\boldsymbol{\Lambda}}_{\text{normalization}} \left(\underbrace{\boldsymbol{A}}_{\substack{\text{adjacency}\\ \text{matrix}}} \odot \underbrace{\boldsymbol{w}\boldsymbol{w}^{\mathsf{T}}}_{\text{weights}} \right) \in \mathbb{R}^{N \times N}$$

$\odot$ Hadamard product

$\boldsymbol{w} \in \mathbb{R}^N$ denotes the weight of each individual's opinion in the network

Adjacency Matrix: Erdős-Rényi Graph

Undirected (eg: Facebook)

$$\boldsymbol{A}_{ij} \begin{cases} \overset{\text{iid}}{\sim} \mathrm{Ber}(p) & \text{for } j < i \\ = 0 & \text{for } j = i \end{cases}$$

$$\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji}, j > i$$

Directed (eg.: Twitter)

$$\boldsymbol{A}_{ij} \begin{cases} \overset{\text{iid}}{\sim} \mathrm{Ber}(p) & \text{for } j \neq i \\ = 0 & \text{for } j = i. \end{cases}$$

Network Opinion Dynamics Model

DeGroot Dynamics Model

Normalized weighted average of neighbors
+
Manipulator's influence

$$\boldsymbol{x}_k = \bar{\boldsymbol{A}}\boldsymbol{x}_{k-1} + \boldsymbol{u}_k \in \mathbb{R}^N$$

$$\bar{\boldsymbol{A}} = \boldsymbol{\Lambda}\left[\boldsymbol{A} \odot \left(\boldsymbol{w}\boldsymbol{w}^{\mathsf{T}}\right)\right]$$

$\boldsymbol{x}_k$: Network opinion at time k

$\boldsymbol{u}_k$: Control input

$\bar{\boldsymbol{A}}$: Weighted adjacency matrix

$s > 0$: Sparsity level

Sparsity Constraint

Manipulator can influence at most s individuals in the network

$$\|\boldsymbol{u}_k\|_0 \leq s$$

$$k = 1, 2, \ldots$$

Non-sparse Vs Sparse

$$\boldsymbol{x}_k = \bar{\boldsymbol{A}}\boldsymbol{x}_{k-1} + \boldsymbol{u}_k \in \mathbb{R}^N$$
$$\bar{\boldsymbol{A}} = \boldsymbol{\Lambda}\left[\boldsymbol{A} \odot \left(\boldsymbol{w}\boldsymbol{w}^{\mathsf{T}}\right)\right]$$

- Note that the input matrix $\boldsymbol{B} = \boldsymbol{I}$
- Classical PBH test:
$$\text{Rank}\left\{\left[\bar{\boldsymbol{A}} - \lambda\boldsymbol{I} \quad \boldsymbol{I}\right]\right\} = N$$
- Non-sparse case: network opinion is always controllable
- Sparse-case: depends on the sparsity bound

- **Evolution model:** $\boldsymbol{x}_k = \boldsymbol{\Lambda}\left(\boldsymbol{A} \odot \left(\boldsymbol{w}\boldsymbol{w}^\mathsf{T}\right)\right)\boldsymbol{x}_{k-1} + \boldsymbol{u}_k$
- **Erdős-Rényi model:** $\boldsymbol{A}_{ij} \begin{cases} \overset{\text{iid}}{\sim} \mathrm{Ber}(p) & \text{for } j < i \\ = 0 & \text{for } j = i \end{cases}$ and $\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji}$
- **Assumption:**

$$(N-s)^{-1} \leq p \leq 1 - (N-s)^{-1}$$

  - Results hold unless p is too small or too large
  - Sparser the system, wider the range of p

Network opinion is controllable with probability at least

$$\sum_{i=0}^{s} \binom{N}{i} (1-p)^{i(2N-i-1)/2} \left[1 - C \exp\left(-c(p(N-i))^{1/32}\right)\right],$$

Network size	N	Sparsity	s
Edge probability	p	Constants	$C, c > 0$

Network opinion is controllable with probability at least

$$\sum_{i=0}^{s} \binom{N}{i} (1-p)^{i(2N-i-1)/2} \left[1 - C \exp\left(-c(p(N-i))^{1/32}\right)\right],$$

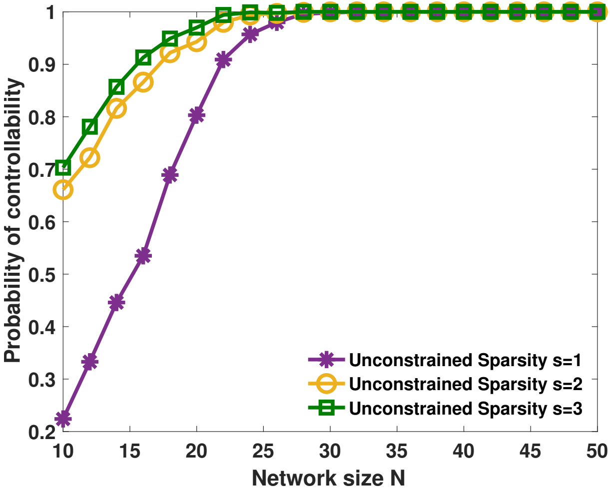
- **Sparsity**: As s increases, more terms in the summation
- **Network size**: As $N \to \infty$, the bound goes to 1

$$\text{bound} \geq 1 - C \exp\left(-c(p(N-i))^{1/32}\right)$$

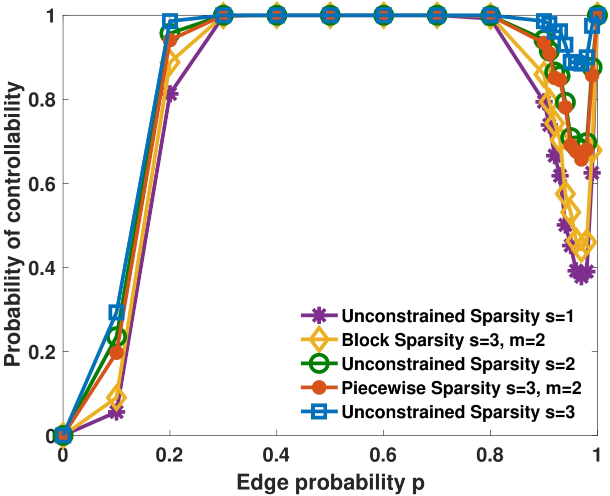
- **Edge probability:**
  - ✓ As $p \to 0$, the bound $\to 0$ ($\boldsymbol{A} \to \mathbf{0}$), and our result holds
  - ✗ As $p \to 1$, the bound $\nrightarrow 1$ ($\boldsymbol{A} \to \mathbf{1}\mathbf{1}^{\mathsf{T}} - \boldsymbol{I}$), and result does not hold

Network size	N	Sparsity	s
Edge probability	p	Constants	$C, c > 0$

Numerical Results: Variation with N and s



Undirected graph with $p = 0.2$



Undirected graph with $N = 20$

Summary of Controllability Results

- **Controllability using sparse inputs:**
  - Polynomial time verification
  - Can be ensured by applying at most N inputs
- **Opinion dynamics:**
  - Unconstrained manipulators succeed with probability one
  - Probability of the manipulator's success increases with network size and sparsity
  - Opinions on an asymptotically large network are almost surely controllable

Design/Estimation of Sparse Control Inputs

Design Problem: Driving the System to a Desired State

- Given an initial state and a final state, how to choose the sparse inputs?
- Recall that we need at most N sparse inputs

$$\underbrace{\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{x}_{\text{init}}}_{\tilde{\boldsymbol{x}} \in \mathbb{R}^{N \times 1}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\boldsymbol{\Phi} \in \mathbb{R}^{N \times NL}} \underbrace{\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_N \end{bmatrix}}_{\boldsymbol{u} \in \mathbb{R}^{NL \times 1}} \implies \tilde{\boldsymbol{x}} = \boldsymbol{\Phi}\boldsymbol{u}$$

Naïve approach:

- Use standard sparse recovery algorithms from CS to estimate $\boldsymbol{u}$
- Need not ensure the sparsity constraints on individual $\boldsymbol{u}_k$'s

- Given an initial state and a final state, how to choose the sparse inputs?
- Recall that we need at most N sparse inputs

$$\underbrace{\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{x}_{\text{init}}}_{\tilde{\boldsymbol{x}} \in \mathbb{R}^{N \times 1}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\boldsymbol{\Phi} \in \mathbb{R}^{N \times NL}} \underbrace{\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_N \end{bmatrix}}_{\boldsymbol{u} \in \mathbb{R}^{NL \times 1}} \implies \tilde{\boldsymbol{x}} = \boldsymbol{\Phi}\boldsymbol{u}$$

Naïve approach:

- Use standard sparse recovery algorithms from CS to estimate $\boldsymbol{u}$
- Need not ensure the sparsity constraints on individual $\boldsymbol{u}_k$'s

Design Problem: Driving the System to a Desired State

- Given an initial state and a final state, how to choose the sparse inputs?
- Recall that we need at most N sparse inputs

$$\underbrace{\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{x}_{\text{init}}}_{\tilde{\boldsymbol{x}} \in \mathbb{R}^{N\times 1}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\boldsymbol{\Phi} \in \mathbb{R}^{N\times NL}} \underbrace{\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_N \end{bmatrix}}_{\boldsymbol{u} \in \mathbb{R}^{NL\times 1}} \implies \tilde{\boldsymbol{x}} = \boldsymbol{\Phi}\boldsymbol{u}$$

Naïve approach:

- Use standard sparse recovery algorithms from CS to estimate $\boldsymbol{u}$
- Need not ensure the sparsity constraints on individual $\boldsymbol{u}_k$'s

Design Problem: Driving the System to a Desired State

- Given an initial state and a final state, how to choose the sparse inputs?
- Recall that we need at most N sparse inputs

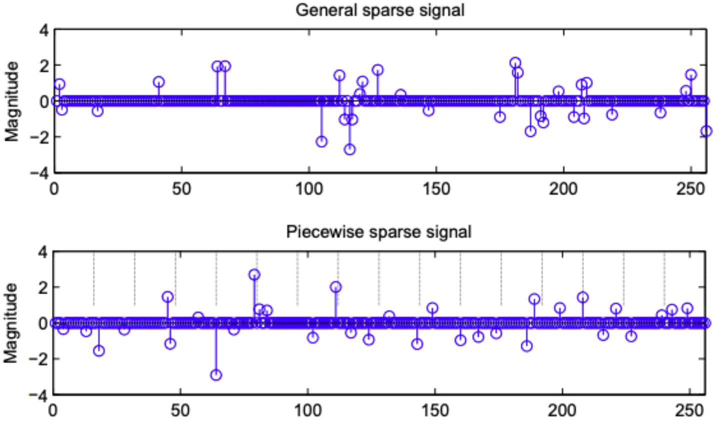
$$\underbrace{\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{x}_{\text{init}}}_{\tilde{\boldsymbol{x}} \in \mathbb{R}^{N \times 1}} = \underbrace{\begin{bmatrix} \boldsymbol{A}^{K-1}\boldsymbol{B} & \boldsymbol{A}^{K-2}\boldsymbol{B} & \ldots \boldsymbol{B} \end{bmatrix}}_{\boldsymbol{\Phi} \in \mathbb{R}^{N \times NL}} \underbrace{\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_N \end{bmatrix}}_{\boldsymbol{u} \in \mathbb{R}^{NL \times 1}} \implies \tilde{\boldsymbol{x}} = \boldsymbol{\Phi}\boldsymbol{u}$$

Naïve approach:

- Use standard sparse recovery algorithms from CS to estimate $\boldsymbol{u}$
- Need not ensure the sparsity constraints on individual $\boldsymbol{u}_k$'s

A Better Approach for Sparse Control Design

- Sparse recovery algorithms specialized for estimating a sparse vector formed by concatenating N sparse vectors
- The new sparsity model is called **piecewise sparsity**

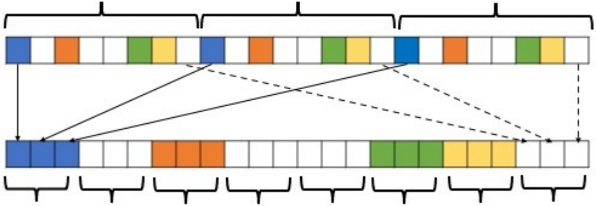


- How to choose the sparse inputs with a common support?

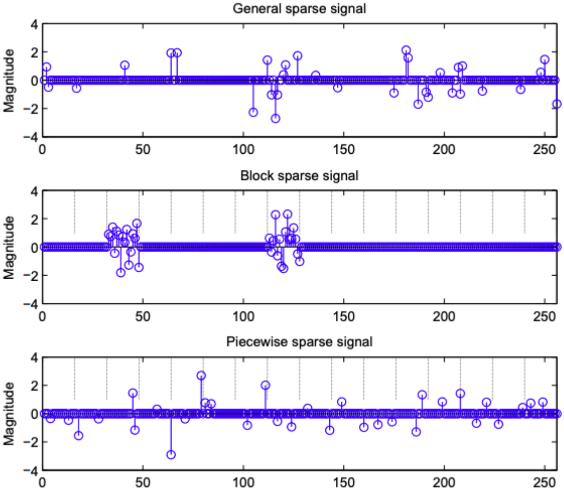
$$\underbrace{\boldsymbol{x}_{\text{final}} - \boldsymbol{A}^K \boldsymbol{x}_{\text{init}}}_{\tilde{\boldsymbol{x}} \in \mathbb{R}^{N\times 1}} = \underbrace{\left[\boldsymbol{A}^{K-1}\boldsymbol{B} \quad \boldsymbol{A}^{K-2}\boldsymbol{B} \quad \ldots \boldsymbol{B}\right]}_{\boldsymbol{\Phi} \in \mathbb{R}^{N\times NL}} \underbrace{\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_N \end{bmatrix}}_{\boldsymbol{u} \in \mathbb{R}^{NL\times 1}} \implies \tilde{\boldsymbol{x}} = \boldsymbol{\Phi}\boldsymbol{u}$$

$\boldsymbol{u}_k$'s have a common support

- Rearrange the sparse vector to get a **block sparse vector**



Different Sparsity Models



- Time-varying support: Limited algorithms
  1. Piecewise OMP
  2. Piecewise inverse scale space algorithm

- Time-invariant support: Plenty of algorithms
  1. Block OMP
  2. Group LASSO
  3. Block SBL
  4. Learned block SBL

     ⋮

Summary

$$\text{State Evolution: } \boldsymbol{x}_k = \boldsymbol{A}\boldsymbol{x}_{k-1} + \boldsymbol{B}\boldsymbol{u}_k$$
$$\text{Observations: } \boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{w}_k$$

Sparsity constraint: More restricted inputs

	Non-sparse	Sparse
Solution	Least-square solution	**Piecewise sparse recovery algorithm** **Block sparse algorithms** **Kalman-SBL algorithms and variants**
Existence	Kalman and PBH test $K = N$	PBH test and $s \geq N - \text{Rank}\{\boldsymbol{A}\}$ $K = N$

PBH = Popov-Belovich-Hautus

Last Slice: Structured LTI system

State Space Model

Dynamics: $\boldsymbol{x}_k = \boldsymbol{A}\boldsymbol{x}_{k-1} + \boldsymbol{B}\boldsymbol{u}_k$

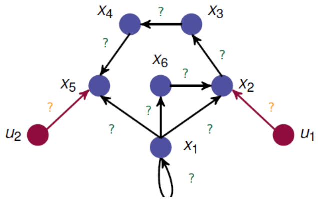
System matrices are known up to their zero/nonzero pattern:

0 $\implies$ edge weight is zero;

x $\implies$ edge weight is non-zero;

? $\implies$ edge weight is zero/non-zero

$$\boldsymbol{x}_k = \begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & ? \\ 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 \\ x & 0 & 0 & ? & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x}_{k-1} + \begin{bmatrix} 0 & 0 \\ x & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & x \\ 0 & 0 \end{bmatrix} \boldsymbol{u}_k$$



A structured LTI system $(\boldsymbol{A}; \boldsymbol{B})$ is called structurally controllable if it is controllable for **at least one** of its numerical realizations $(A; B)$

A structured LTI system $(\boldsymbol{A}; \boldsymbol{B})$ is called strong structurally controllable if it is controllable for **all** of its numerical realizations $(A; B)$

- Structural controllability is defined using zero-forcing numbers in graphs
- Instead of Kalman and PBH test, we have **graph coloring based tests**
- Similar problems can be explored for structured systems

Open Problems

- Other types of sparsity constraints: sparsity over time, actuator use, feedback control
- New algorithms design/estimation? Do deep learning-based algorithms help?
- New applications related to networks? Graphical models?
- Sparsity + structural controllability is almost untouched

References - Our Work

- G. Joseph and C. R. Murthy, Sparsity-Constrained Linear Dynamical Systems, Springer, 2024.
- G. Joseph, Sparse Actuator Control of Discrete-Time Linear Dynamical Systems, Foundations and Trends in Systems and Control, vol. 11, no. 3, pp 186–284, 2024.
- L. Ballotta, G. Joseph, I. R. Thete, "Pointwise-Sparse Actuator Scheduling for Linear Systems with Controllability Guarantee," in IEEE Control Systems Letters, vol. 8, pp. 2361-2366, 2024
- G. Joseph, Output Controllability of a Linear Dynamical System with Sparse Controls, in IEEE Transactions on Control of Network Systems, 2022.
- G. Joseph, Controllability of a Linear System with Nonnegative Sparse Controls, IEEE Transactions on Automatic Control, vol. 67, no. 1, pp. 468-473, Jan. 2022.
- G. Joseph, B. Nettasinghe, V. Krishnamurthy, and P. K. Varshney, Controllability of Network Opinion in Erdos-Renyi Graphs using Sparse Control Inputs, in SIAM Journal on Control and Optimization, vol. 59, no. 3, pp. 2321–2345, 2021.
- G. Joseph and C. R. Murthy, Controllability of Linear Dynamical Systems Under Input Sparsity Constraints, in IEEE Transactions on Automatic Control, vol. 66, no. 2, pp. 924-931, Feb. 2021.
- K. Kondapi, C. . Sriram, G. Joseph, and C. R. Murthy, Sparse Actuator Scheduling for Discrete-Time Linear Dynamical Systems, in Proc. Indian Control Conference, Bhopal, Dec. 2024.
- G. Joseph, S. Moothedath, and J. Lin, Minimal Input Structural Modifications for Strongly Structural Controllability, in Proc. CDC, Milan, Italy, Dec. 2024.
- C. Sriram, G. Joseph, and C. R. Murthy, Control of Linear Dynamical Systems Using Sparse Inputs, in Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), Barcelona, Spain, May 2020.

Acknowledgements

- Chandra R. Murthy, IISc Bangalore
- Chandrasekhar Sriram, IISc Bangalore
- Rupam Kalyan Chakraborty, IISc Bangalore

- Pramod K. Varshney, Syracuse University
- Vikram Krishnamurthy, Cornell University
- Buddhika Nettasinghe, Cornell University
- Shana Moothedath, Iowa Sate

- Luca Ballota, TU Delft
- Irawati Thete, TU Delft

Geethu Joseph
Signal Processing Systems
Faculty of Electrical Engineering, Mathematics, and Computer Science
Email: g.joseph@tudelft.nl