Robust Covariance Neural Networks

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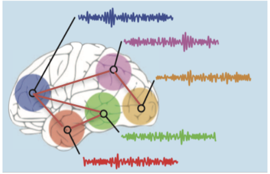
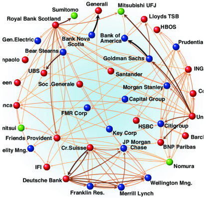
Covariance relationships between data points

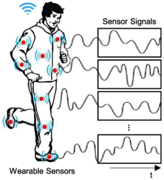
- Data usually contains latent interconnections



Brain



Financial



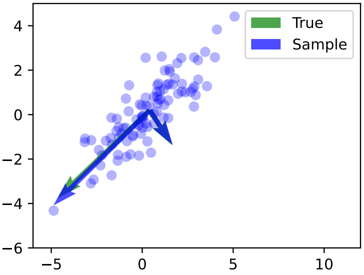
Body movement

- One way to capture these relations is through the covariance matrix

  $\Rightarrow \mathbf{C} = \mathbb{E}[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^{\mathsf{T}}]$

Principal Component Analysis (PCA)

- Project the data onto the covariance eigenspace
  - ⇒ $\mathbf{C} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^\top$
  - ⇒ $\tilde{\mathbf{x}} = \mathbf{V}^\top\mathbf{x}$
- Select (filter) the directions that maximize the variance of data points
  - ⇒ Used for dimensionality reduction by selecting only a few eigenvectors



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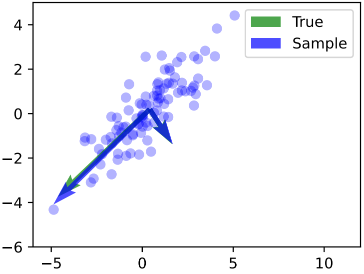

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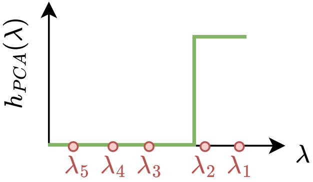
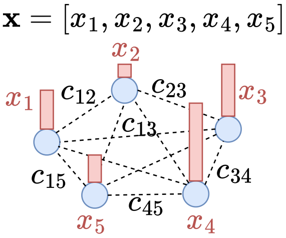


PCA filtering

Covariance Filters

- Data sample $\mathbf{x} = [x_1, \ldots, x_N]^\mathsf{T}$ with covariance $\mathbf{C}$
  - ⇒ Build a graph where:
  - ⇒ the features are the node signals x_i
  - ⇒ the edges are the covariance values c_{ij} → fully-connected graph



Covariance Filters

- Definition: Graph convolution covariance filters

$$\mathbf{z} = \mathbf{H}(\hat{\mathbf{C}})\mathbf{x} = \sum_{k=0}^{K} h_k \hat{\mathbf{C}}^k \mathbf{x}$$

⇒ learnable parameters: h_k

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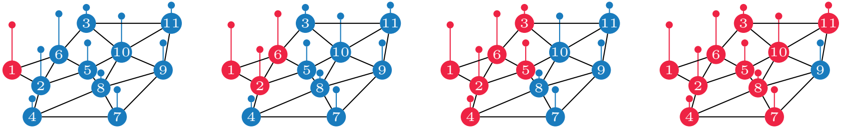
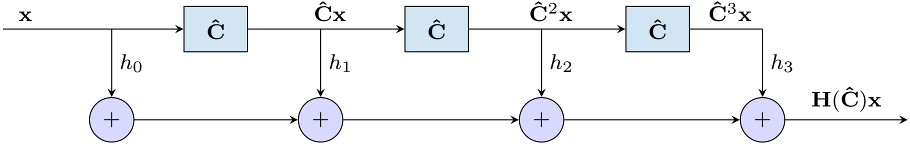


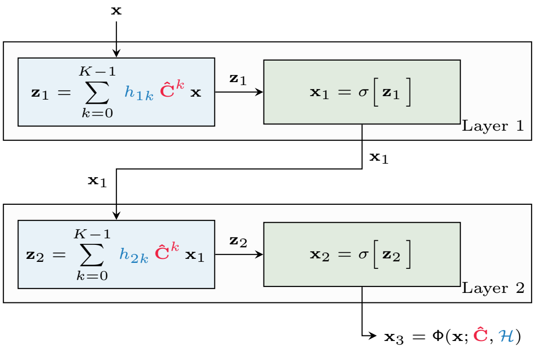


- $\hat{\mathbf{C}}^k\mathbf{x}$ shifts signal $\mathbf{x}$ k times over the covariance graph $\hat{\mathbf{C}}$

Covariance Neural Networks (VNNs)

- Definition: Covariance filters followed by pointwise nonlinearities σ

$$\mathbf{x}^l = \sigma\left(\mathbf{H}^l(\hat{\mathbf{C}})\mathbf{x}^{l-1}\right) \quad l = 1, \ldots, L.$$



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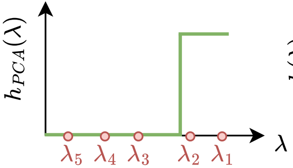
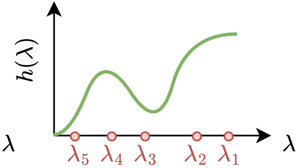
$\Rightarrow$ The covariance filter processes the principal components $\hat{\mathbf{V}}^\top \mathbf{x}$!





Outliers

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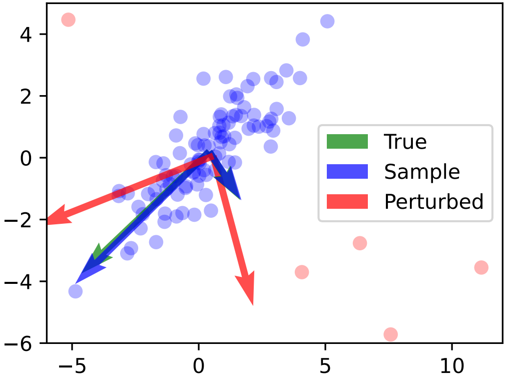
  - ⇒ PCA estimation is heavily affected
  - ⇒ VNNs do not work reliably



Robust Covariance Neural Networks

Desiderata:

- Use covariances for data processing
  - ⇒ with a learnable filter function
- Be robust to outliers and missing values
  - ⇒ via covariance correction terms learned end-to-end
- Be stable to finite-sample estimation errors

Outline

- Robust Covariance Neural Networks
- Theoretical results
- Experiments
- Conclusions

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- **Robust Covariance Neural Networks**
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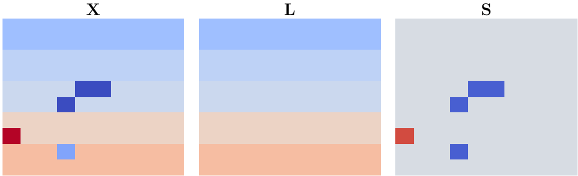
Data Perturbation Model

- Following robust PCA:
  - ⇒ Data $\mathbf{L}$ is low-rank
  - ⇒ Perturbations $\mathbf{S}$ are sparse

$$\mathbf{X} = \mathbf{L} + \mathbf{S}$$



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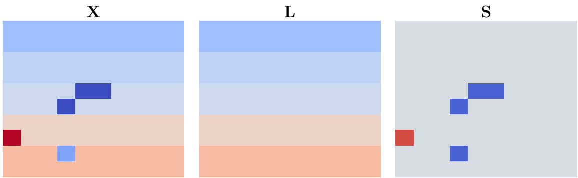

- The observed covariance $\hat{\mathbf{C}}$ is

$$\hat{\mathbf{C}} = \mathbf{X}\mathbf{X}^\top / T = (\mathbf{L}+\mathbf{S})(\mathbf{L}+\mathbf{S})^\top / T = (\mathbf{L}\mathbf{L}^\top + \mathbf{S}\mathbf{L}^\top + \mathbf{L}\mathbf{S}^\top + \mathbf{S}\mathbf{S}^\top)/T$$

⇒ where $\mathbf{S}\mathbf{L}^\top + \mathbf{L}\mathbf{S}^\top$ is low-rank and $\mathbf{S}\mathbf{S}^\top$ is sparse

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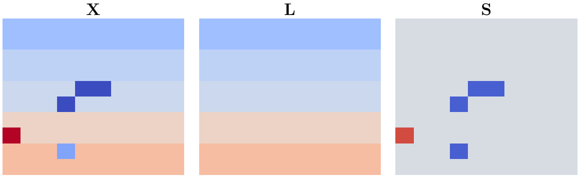

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  - ⇒ where $\mathbf{S}\mathbf{L}^\intercal + \mathbf{L}\mathbf{S}^\intercal$ is low-rank and $\mathbf{S}\mathbf{S}^\intercal$ is sparse
- We reconstruct the clean covariance $\hat{\mathbf{C}} = \mathbf{L}\mathbf{L}^\intercal/T$ as

$$\hat{\mathbf{C}} = \hat{\mathbf{C}} + \mathbf{E}_s + \mathbf{E}_l \quad (1)$$

  - ⇒ where $\mathbf{E}_s$ is sparse and $\mathbf{E}_l$ is low-rank.

Robust Covariance Neural Networks (RVNNs)

- RVNNs are VNNs Φ trained with the following objective

$$\min_{\mathcal{H},\mathbf{E}_s,\mathbf{E}_l} \mathcal{L}(\Phi(\mathcal{H}, \hat{\mathbf{C}} + \mathbf{E}_s + \mathbf{E}_l), \mathbf{X}_{\text{tr}}, \mathbf{y}_{\text{tr}}) + \gamma_s \|\mathbf{E}_s\|_1 + \gamma_l \|\mathbf{E}_l\|_*$$

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⇒ promote low-rank structure for $\mathbf{E}_l$ via the nuclear norm

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Stability of RVNNs

- The observed covariance $\hat{\mathbf{C}}$ contains multiple perturbations w.r.t. the true covariance $\mathbf{C}$

  $\Rightarrow$ due to outliers

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- Stability of RVNN to covariance perturbations

$$\|\mathbf{H}(\hat{\mathbf{C}} + \mathbf{E}_s + \mathbf{E}_l) - \mathbf{H}(\mathbf{C})\| \leq P\sqrt{N}(1 + \sqrt{N})(\mathcal{O}(T^{-1/2}) + \delta)$$

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  - ⇒ Stability improves with increasing number of samples T
  - ⇒ $\delta = \|\tilde{\mathbf{C}} - \hat{\mathbf{C}} - \mathbf{E}_s - \mathbf{E}_l\|$ measures the quality of reconstruction of the clean covariance matrix

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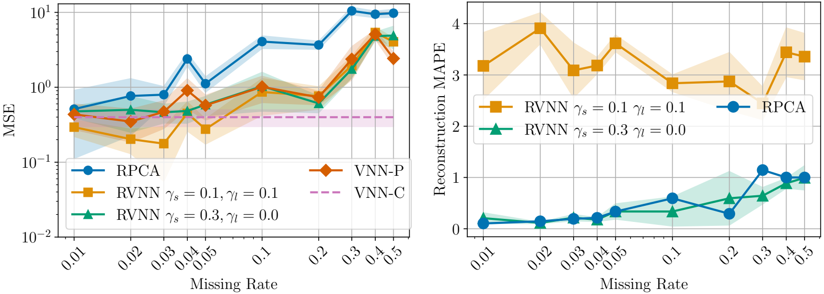
Synthetic Dataset

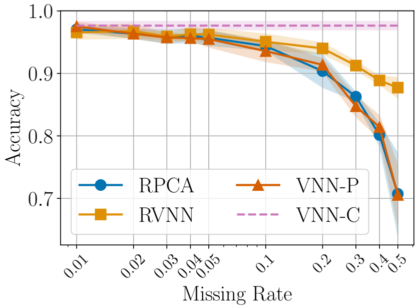
- Setup: Regression task, varying size of missing data
- Baselines: VNN on clean data (VNN-C), VNN on perturbed data (VNN-P), RPCA+VNN

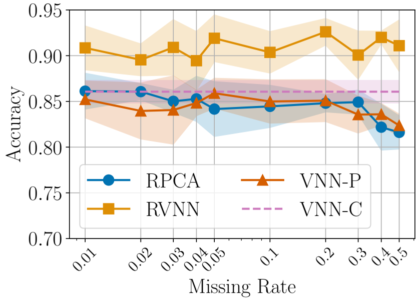


- Results:
  - ⇒ RVNN matches or improves performance of VNN-C for missing rate ≤ 0.05
  - ⇒ Based on γ_s, γ_l, covariance reconstruction is good or bad

Real Datasets

- **Datasets**:
  - ⇒ Brain recordings before and after epilepsy seizure – binary classification
  - ⇒ Motion sensor recordings – activity classification





- **Results**:
  - ⇒ RVNN less affected by missing values than other models

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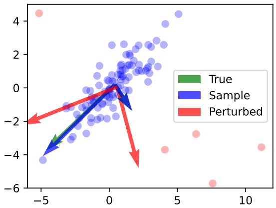
- Outliers and missing values make covariance estimation difficult

  $\Rightarrow$ PCA and VNN are unreliable



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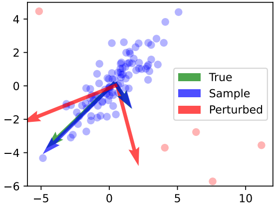

- Robust Covariance Neural Networks (RVNNs)

  $\Rightarrow$ Sparse and low-rank covariance correction learned end-to-end

  $\Rightarrow$ Stable to finite-sample estimation errors

  $\Rightarrow$ Maintain downstream performance for varying missing rates

$$\|\mathbf{H}(\hat{\mathbf{C}} + \mathbf{E}_s + \mathbf{E}_l) - \mathbf{H}(\mathbf{C})\| \leq P\sqrt{N}(1+\sqrt{N})(\mathcal{O}(T^{-1/2}) + \delta)$$

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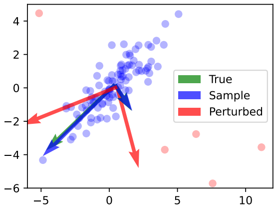

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- Thank you for the attention!
  - $\Rightarrow$ a.cavallo@tudelft.nl